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CBSE

CLASS 12TH

THE CENTRAL BOARD OF SECONDARY EDUCATION

PART – VII

MATHEMATICS - II

SIERRA INNOVATIONS

MATHEMATICS - 2

CHAPTER-8 APPLICATION OF INTEGRALS

Content:

Topics:

- Introduction
- Area under simple curve
  - The area of the region bounded by a curve and line
- Area between two curves
- Questions and Answers

Introduction

In this chapter we are going to deal with areas enclosed by curves area between lines and arcs of circles, parabolas and ellipses (standard forms only) using integral technique:

Area under Simple Curve:

Consider the figure below, we can think of area under the curve as composed of large number of very thin vertical strips. Consider
an arbitrary strip of height *y* and width *dx*, then *d*A (area of the elementary strip)= *ydx*, where, *y* = *f*(*x*).
This area is called the *elementary area* which is located at an arbitrary position within the region which is specified by some value of *x* between *a* and *b.*

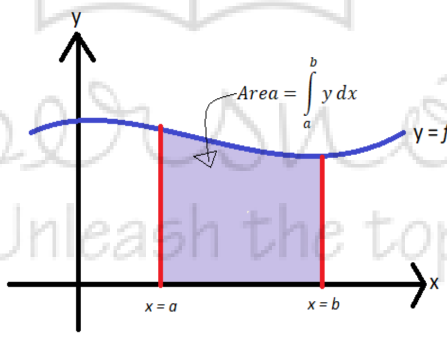


Fig.1a

(i) Total area bounded by the curve $y = f(x)$, between the ordinates $x = a$ and $x = b$ (Fig.1a) can be found by using definite integrals and represented as

$$Area = \int_a^b y\,dx$$

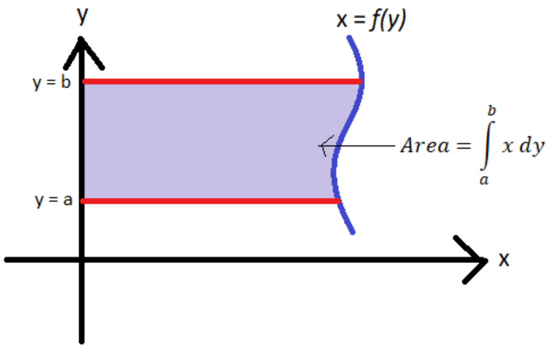


Fig.1b

ii.If the curve is given as $x = g(y)$ (Fig.1b) , then the area bounded by the given curve between $y = a$ and $y = b$ $(b > a)$ can be represented as

$$Area = \int_a^b x\,dy$$

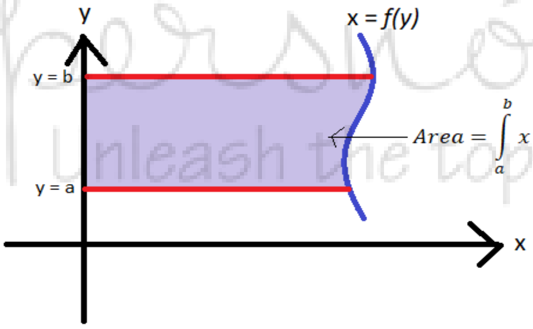


Fig. 1c

iii. If the curve is given as $y = f(x)$ (Fig.1c) and $f(x) < 0$, then the area bounded by the given curve between $x = a$ and $x = b$ $(b > a)$ can be represented as

$$Area = \left| \int_a^b f(x)dx \right|$$

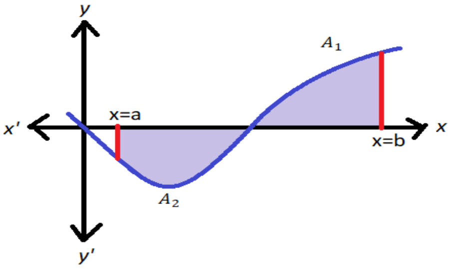


Fig.1d

iv. If the curve is given as $y = f(x)$ (Fig.1d) and some portion of curve lies above the x-axis and some below it such that $A1 < 0$ and $A2 > 0$, then the area bounded by the given curve between $x = a$ and $x = b\ (b > a)$ can be represented as

Area= |A1|+A2

Example: Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution: Here we have,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or, $\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$

or, $\frac{y^2}{b^2} = \frac{a^2 - x^2}{a^2}$

or, $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$

or, $y = \pm\sqrt{\frac{b^2}{a^2}(a^2 - x^2)}$

or, $y = \pm\frac{b}{a}\sqrt{(a^2 - x^2)}$

We know that,

Ellipse is symmetrical about x-axis and y-axis.

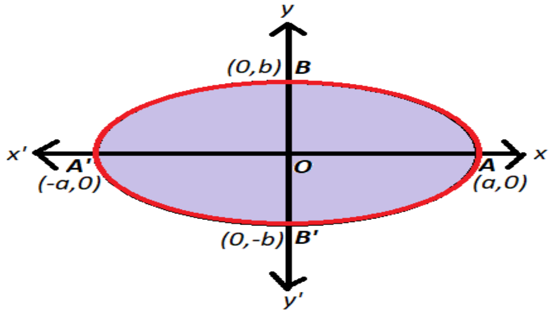


Fig.2a

Area of ellipse = 4 × $\int_0^a y\,dx$

$= 4\int_0^a \frac{b}{a}\sqrt{(a^2 - x^2)}dx$

$= \frac{4b}{a}\int_0^a \sqrt{(a^2 - x^2)}\,dx$

$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{(a^2 - x^2)} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$

$= \frac{4b}{a}\left[\left(\frac{a}{2}\sqrt{a^2 - a^2} + \frac{a^2}{2}\sin^{-1}\frac{a}{a}\right) - \left(\frac{0}{2}\sqrt{a^2 - 0} + \frac{a^2}{2}\sin^{-1}(0)\right)\right]$

$= \frac{4b}{a}\left[0 + \frac{a^2}{2}\sin^{-1}(1) - 0 - 0\right]$

$= \frac{4b}{2} \times \frac{a^2}{2}\sin^{-1}(1)$

= 2ab × $\sin^{-1}(1)$

= 2ab × $\pi/2$

= πab

The area of the region bounded by a curve and a line:

We will find the area of the region bounded by a line and a circle, a line and a parabola, a line and an ellipse.

Example: Find the area of the region bounded by the parabola $y^2 = 2px$, $x^2 = 2py$.

Solution:

Given that, parabolas are $y^2 = 2px$ .... (i)

and $x^2 = 2py$ .... (ii)

Now, from equation (ii) we have

y = x2 /2p

Putting the value of y in equation (i), we have

(x2 /2p)2 = 2*px*

x4 /4p2 = 2px

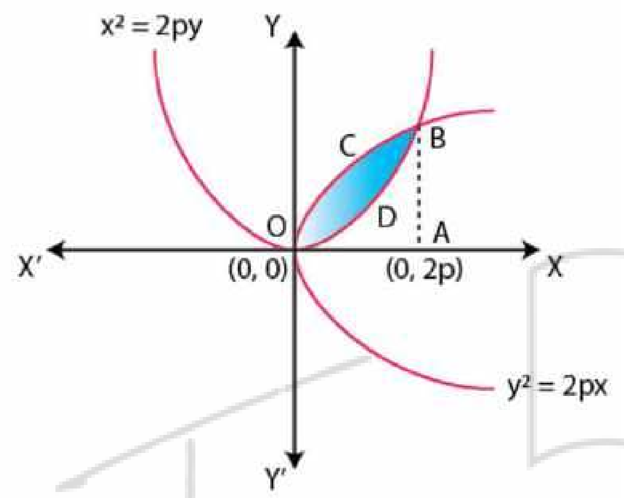
x4 = 8p3x

x4 – 8p3 x = 0

x(x3 – 8p3) = 0

So, x = 0 or x3 – 8p3 = 0 ⇒ x = 2p

Now, the required area is



= Area of the region (OCBA – ODBA)

$$= \int_0^{2p} \sqrt{2px}\, dx - \int_0^{2p} \frac{x^2}{2p} dx = \sqrt{2p} \int_0^{2p} \sqrt{x}\, dx - \frac{1}{2p} \int_0^{2p} x^2 dx$$

$$= \sqrt{2p} \cdot \frac{2}{3} [x^{3/2}]_0^{2p} - \frac{1}{2p} \cdot \frac{1}{3} [x^3]_0^{2p}$$

$$= \frac{2\sqrt{2}}{3} \sqrt{p}\, [(2p)^{3/2} - 0] - \frac{1}{6p} [(2p)^3 - 0]$$

$$= \frac{2\sqrt{2}}{3} \sqrt{p} \cdot 2\sqrt{2}\, p^{\frac{3}{2}} - \frac{1}{6p} \cdot 8p^3$$

$$= \frac{8}{3} \cdot p^2 - \frac{8}{6} p^2 = \frac{8}{6} p^2 = \frac{4}{3} p^2 \text{ sq. units}$$

Thus, the required area is 4/3 p2 sq. units.

Example: Find the area of the region bounded by the curve $y = x^3$ and $y = x + 6$ and $x = 0$.

Solution:

Given that curves are $y = x^3$, $y = x + 6$ and $x = 0$

On solving $y = x^3$ and $y = x + 6$, we get,

$x^3 = x + 6$

$x^3 - x - 6 = 0$

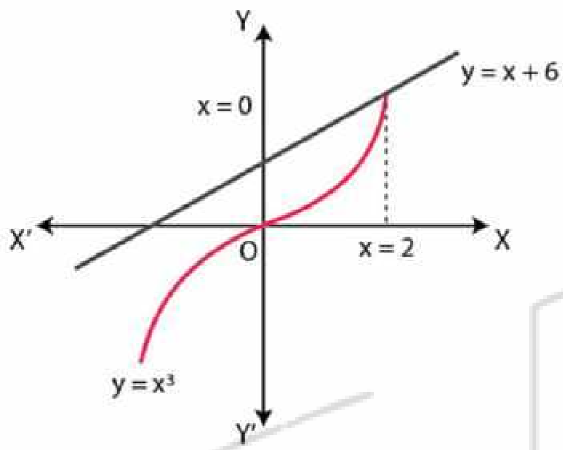
$x^2 (x - 2) + 2x(x - 2) + 3(x - 2) = 0$

(x - 2) (x2 + 2x + 3) = 0

It's seen that $x^2 + 2x + 3 = 0$ has no real roots

So, x = 2 is the only root for the above equation.

So, the required area of the shaded region is given by



$$= \int_0^2 (x+6)\,dx - \int_0^2 x^3\,dx$$

$$= \left[\frac{x^2}{2} + 6x\right]_0^2 - \frac{1}{4}[x^4]_0^2$$

$$= \left(\frac{4}{2} + 12\right) - (0+0) - \frac{1}{4}[(2)^4 - 0]$$

$$= 14 - \frac{1}{4} \times 16 = 14 - 4 = 10 \text{ sq. units.}$$

Area between Two Curves:

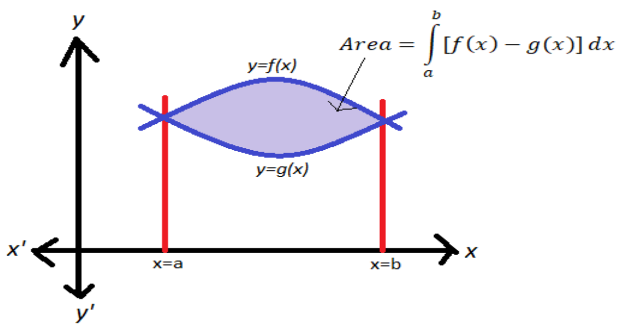


Fig.2a

i. If two curves are given as $y = f(x)$ and $y = g(x)$,where $f(x) \geq g(x)$ in $[a, b]$ (Fig 8.5), then

$$Area = \int_a^b [f(x) - g(x)]\,dx$$

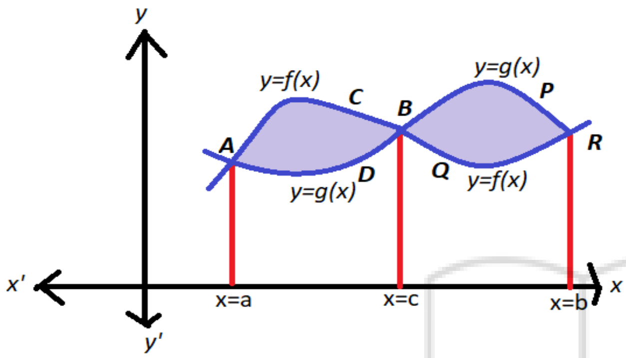


Fig. 2b

ii. If $f(x) \geq g(x)$ in $[a, c]$ and $g(x) \geq f(x)$ in $[c, b]$ (Fig 8.6), then total area A can be given as

Total Area = Area of the region ACBDA + Area of the region BPRQB

$$A = \int_a^c [f(x) - g(x)]dx + \int_c^b [g(x) - f(x)]dx$$

Example: Find the area of the region bounded by the curves $y^2 = 9x$, $y = 3x$.

Solution:

Given curves are $y^2 = 9x$ and $y = 3x$

Now, solving the two equations we have

$(3x)^2 = 9x$

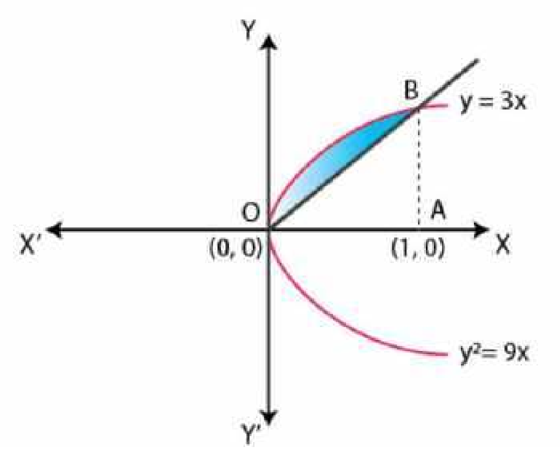
$9x^2 = 9x$

$9x^2 - 9x = 0 \Rightarrow 9x(x - 1) = 0$

Thus, x = 0, 1

So, the area of the shaded region is given by

= ar(region OAB) – ar (Δ OAB)



$= \text{ar (region OAB)} - \text{ar }(\Delta OAB)$

$= -\int_0^1 y_1 \cdot dx = \int_0^1 \sqrt{9x}\, dx - \int_0^1 3x\, dx$

$= 3\int_0^1 \sqrt{x}\, dx - 3\int_0^1 x\, dx = 3\times\frac{2}{3}\left[x^{3/2}\right]_0^1 - 3\left[\frac{x^2}{2}\right]_0^1$

$= 2\left[(1)^{3/2} - 0\right] - \frac{3}{2}\left[(1)^2 - 0\right] = 2(1) - \frac{3}{2}(1) = 2 - \frac{3}{2} = \frac{1}{2}$ sq. units

Thus, the required area is ½ sq.units.

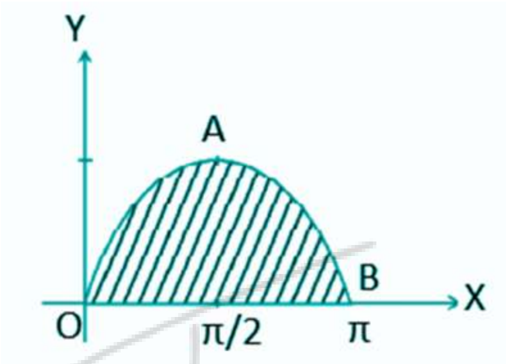
Questions and Answers:

1 Mark each:

1. Find the area bounded by the curve y = sin x between 0 and π.

Solution:

Here we have, curve y = sin x



Area bounded by curve OAB

$$= \int_0^\pi y\,dx$$

$$= \int_0^\pi \sin x\,dx$$

$$= [-\cos x]_0^\pi$$

= – [cos π – cos 0]

= -(-1 -1)

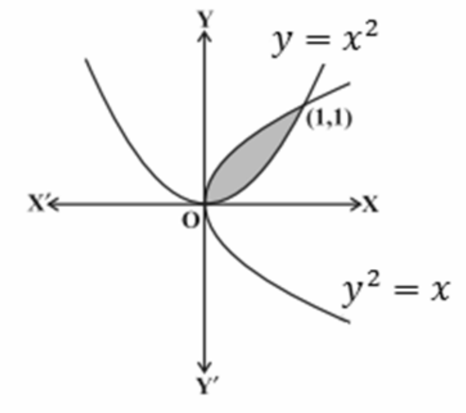
= 2 square units

2. Find the area of the region bounded by the two parabolas $y = x^2$ and $y^2 = x$.

Solution:

We have, two parabolas $y = x^2$ and $y^2 = x$.

The point of intersection of these two parabolas is O (0, 0) and A (1, 1) as shown in the figure below



And,

$y^2 = x$

$y = \sqrt{x} = f(x)$

$y = x^2 = g(x)$, where, $f(x) \geq g(x)$ in $[0, 1]$.

Area of the shaded region

$$= \int_0^1 [f(x) - g(x)]dx$$

$$= \int_0^1 [\sqrt{x} - x^2]dx$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1$$

$= (2/3) - (1/3)$

$= 1/3$

Therefore, the required area is $1/3$ square units.

3. Find the area enclosed by the ellipse $x^2/a^2 + y^2/b^2 = 1$.

Solution:

Here, we have,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\frac{y^2}{b^2} = \frac{a^2 - x^2}{a^2}$$

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$y = \pm\sqrt{\frac{b^2}{a^2}(a^2 - x^2)}$$

$$y = \pm\frac{b}{a}\sqrt{(a^2 - x^2)}$$

And, We know that, Ellipse is symmetrical about both x-axis and y-axis.

So, Area of ellipse = 4 × Area of AOB

$= 4 \times \int_0^a y\,dx$

Substituting the positive value of y in the above expression since OAB lies in the first quadrant.

$$= 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}dx$$

$$= \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}dx$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{4b}{a}\left[\left(\frac{a}{2}\sqrt{a^2 - a^2} + \frac{a^2}{2}\sin^{-1}\frac{a}{a}\right) - \left(\frac{0}{2}\sqrt{a^2 - 0} - \frac{a^2}{2}\sin^{-1}(0)\right)\right]$$

$$= \frac{4b}{a}\left[0 + \frac{a^2}{2}\sin^{-1}(1) - 0 - 0\right]$$

$$= \frac{4b}{a} \times \frac{a^2}{2}\sin^{-1}(1)$$

= 2ab × sin-1(1)

= 2ab × π/2

= πab

4. Find the area bounded by the y-axis, y = cos x and y= sin x when $0 \le x \le \frac{\pi}{2}$

Solution:

Graph of both the functions will intersect at the point

B $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$.

So, the required Shaded Area=

$$\int_0^{\pi/2} \cos x\, dx - \left(\int_0^{\pi/4} \sin x\, dx + \int_{\pi/4}^{\pi/2} \cos x\, dx \right)$$

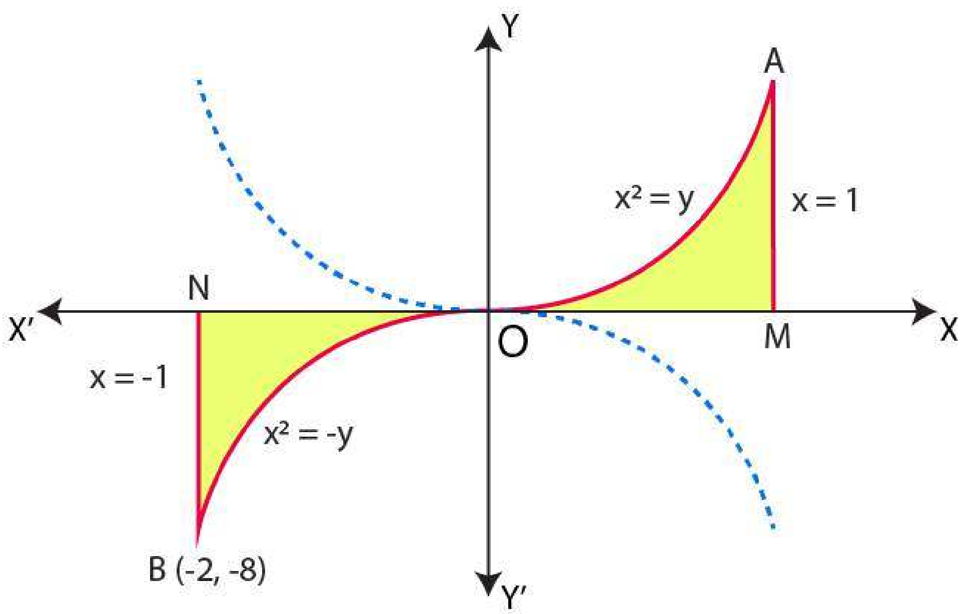
$$\left(\sin\frac{\pi}{2} - \sin 0^\circ\right) - \left(-\cos\frac{\pi}{4} + \cos 0^\circ + \sin\frac{\pi}{2} - \sin\frac{\pi}{4}\right)$$

$$= 1 + \frac{1}{\sqrt{2}} - 1 - 1 + \frac{1}{\sqrt{2}}$$

$\left(\sqrt{2} - 1\right)$ square units.

5. Find the area bounded by the curve y = x|x|, x- axis and the ordinates x = -1 and x = 1

Equation of the curve is



. $y = x|x| = x(x) = x^2$ if $x \geq 0$ ....(1)

And, $y = x|x| = x(-x) = -x^2$ if $x \leq 0$ ....(2)

Required area = Area ONBO + Area OAMO

$$\int_{-1}^{0} -x^2\, dx + \int_{0}^{1} x^2\, dx$$

= 2/3 sq. units

6. Find the area of the circle $x^2 + y^2 = 16$ exterior to the parabola $y^2 = 6x$.

Solution:

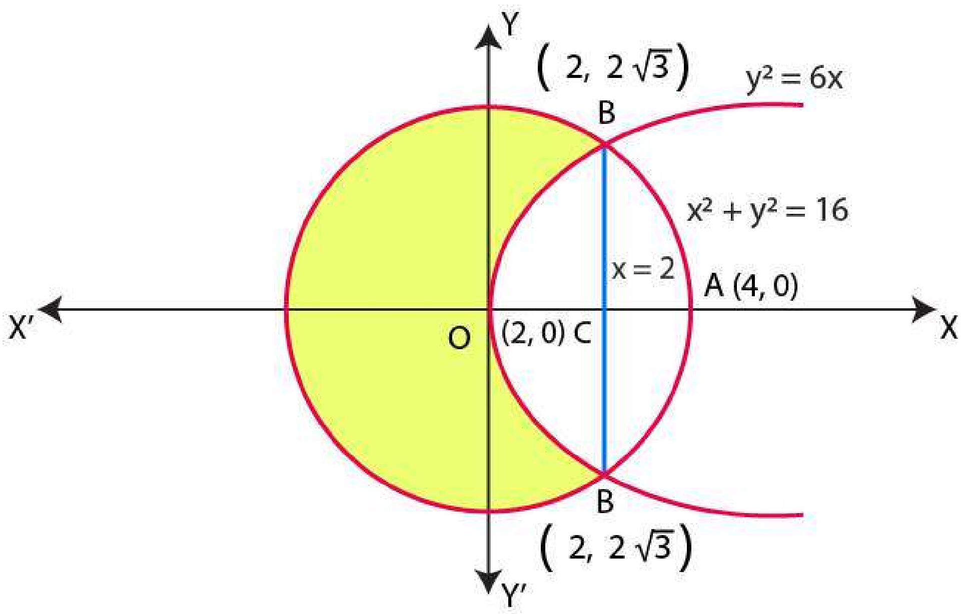
We know, Equation of the circle is $x^2 + y^2 = 16$ .....(1)

Thus, radius of circle is 4

This circle is symmetrical about x-axis and y- axis.

Here two points of intersection are B $(2, 2\sqrt{3})$

and B' $(2, -2\sqrt{3})$.



Required area = Area of circle – Area of circle interior to the parabola

= πr^2 - Area OBAB'O

= 16π - 2 x Area OBACO

= 16π - 2[Area OBCO + Area BACB]

$$16\pi-2\left[\int_0^2 \sqrt{6x}\,dx+\int_2^4 \sqrt{16-x^2}\,dx\right]$$

$$16\pi-2\left[\frac{2}{3}\sqrt{6}\left(2\sqrt{2}\right)+8\sin^{-1}1-\sqrt{12}-8\sin^{-1}\frac{1}{2}\right]$$

$$16\pi-2\left[\frac{8}{\sqrt{3}}+8.\frac{\pi}{2}-2\sqrt{3}-8.\frac{\pi}{6}\right]$$

$$16\pi-2\left[\frac{8}{\sqrt{3}}-2\sqrt{3}+8\pi\left(\frac{1}{2}-\frac{1}{6}\right)\right]$$

$$16\pi-2\left[\frac{2}{\sqrt{3}}+\frac{8\pi}{3}\right]$$

$\frac{4}{3}\left(8\pi-\sqrt{3}\right)$ square units.

7. Find the area of the region bounded by *y* = √*x* and *y* = *x*.

Solution:

Given that, equations of curve *y* = √*x* and line y = x

Solving the equations *y* = √*x* ⇒ y2 = x and y = x, we get

x2 = x

x2 – x = 0

x(x - 1) = 0

So,

x = 0, 1

Now, the required area of the shaded region